Corollaries of the Three Principles

Cor 1. For $E \subseteq \mathbb{R}$, TFSAE (the FSAE):

(i) $E \in \mathfrak{M}$ and $m(E) < +\infty$

(ii) $\forall \varepsilon > 0$, $\exists\ \mathcal{U} = \bigcup_{i=1}^{n} I_i$, disjoint open intervals of finite lengths, such that $m^*(E \triangle \mathcal{U}) < \varepsilon$.

(iii) $\forall \varepsilon > 0$, $\exists\ \mathcal{U} = \bigcup_{i=1}^{n} I_i$, disjoint open intervals of finite lengths s.t.

$$\chi_E = \chi_{\mathcal{U}} \text{ on } \mathbb{R} \setminus A \text{ for some } A \text{ with } m^*(A) < \varepsilon.$$

(one can use m in place m^* in (ii) & (iii))

Proof. (ii)$\Rightarrow$(i): Since $m^*(E \triangle \mathcal{U}) = \inf\{m(O) : \text{open } O \supseteq E \triangle \mathcal{U}\}$, (ii) implies that $\exists$ open $O_\varepsilon \supseteq E \triangle \mathcal{U}$ with $m(O_\varepsilon) < \varepsilon$. Then

$$(O_\varepsilon \cup \mathcal{U}) \setminus E \supseteq O_\varepsilon \cup (\mathcal{U} \setminus E) \text{ of mea} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and $O_\varepsilon \cup \mathcal{U}$ is open and covers E. Hence E is outer-regular so it follows from the 1st principle that and $m^*(E) = m(E) \leq m(O_\varepsilon \cup \mathcal{U}) < +\infty$.

(i)$\Rightarrow$(ii): See the 1st principle (basically use the outer-regularity theorem and the structure of open sets).

(ii)$\Leftrightarrow$(iii): Exercise

Corollary 2. Let $E \in \mathfrak{M}$ with $m(E) < +\infty$ and let f be a simple function vanishing outside E: $f(x) = 0\ \forall x \notin E$. Then $\forall \varepsilon > 0$ $\exists$ a step function $g : \mathbb{R} \to \mathbb{R}$ vanishing outside a finite interval such that $f = g$ on $\mathbb{R} \setminus A$ with some A of measure $< \varepsilon$.

Proof. By assumption, $f = \sum_{i=1}^{N} a_i \chi_{E_i}$ with $a_i \in \mathbb{R}$ & $E_i \in \mathfrak{M}$, $E_i \subseteq E\ \forall i$ (so $m(E_i) < +\infty$). Applying Cor 1 to each E_i to obtain $\mathcal{U}_i$ representable as a union of finitely many open intervals of finite length such that $m(E_i \triangle \mathcal{U}_i) < \frac{\varepsilon}{N}$. Then $g := \sum_{i=1}^{N} a_i \chi_{\mathcal{U}_i}$ is a step-function vanishing outside a

finite interval such that
$$f = g \text{ on } \mathbb{R} \setminus \bigcup_{i=1}^{N} (E_i \triangle U_i)$$
with the union is of measure $< N \cdot \frac{\varepsilon}{N} = \varepsilon$.

Corollary 3. Let $E \in \mathfrak{M}$ and $m(E) < +\infty$; let $f: E \to \overline{\mathbb{R}} = [-\infty, \infty]$ be a measurable function and $f(x) \in \mathbb{R}$ a.e. on E. Let $\varepsilon > 0$. Then $\exists\, \varphi, \psi, g$ (respectively simple, step, continuous functions on $\mathbb{R}$) vanishing outside a finite interval such that

(*) $|f-\varphi|, |f-\psi|, |f-g| < \varepsilon$ on $E \setminus A$ for some A with $m(A) < \varepsilon$

Proof. For each $n \in \mathbb{N}$ let
$$E_n := \{x \in E \cap [-n, n] : |f(x)| < n\} \quad (\in \mathfrak{M},\ E_n \subseteq E_{n+1}\ \forall n)$$
Then $\bigcup_{n \in \mathbb{N}} E_n$ and E are of the same measure, so (why?) $\exists\, N \in \mathbb{N}$ s.t. $A_1 := E \setminus E_N$ is of measure $< \varepsilon/4$. Now, as $-N < f(x) < N\ \forall x \in E_N$ there exists (why) a simple function φ on $\mathbb{R}$ vanishing outside E_N ($\subseteq [-N, N]$) such that
$$|f - \varphi| < \varepsilon \text{ on } E_N (= E \setminus A_1) \quad (1)$$
For this φ, $\exists$ a step-function ψ vanishing outside $[-N, N]$ such that (why)
$$|\varphi - \psi| = 0 \text{ on } \mathbb{R} \setminus A_2 \text{ for some } A_2 \text{ with } m(A_2) < \frac{\varepsilon}{3} \quad (2)$$
For this step ψ, $\exists$ a continuous g vanishing outside $[-N, N]$ such that (please supply details) such that
$$|\psi - g| = 0 \quad \mathbb{R} \setminus A_3 \text{ for some } A_3 \text{ with } m(A_3) < \frac{\varepsilon}{3} \quad (3)$$

Combining (1), (2) and (3), and letting $A = A_1 \cup A_2 \cup A_3$ one has

$$|f - g| < \varepsilon \text{ on } E \setminus A, \text{ and } m(A) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

(and, of course, also $|f - \psi| < \varepsilon$ on $E \setminus A$).

Corollary 4. Let E, f be as in Cor 3. Then $\exists$ sequences (φ_n), (ψ_n), (g_n) (resp. simple, step, continuous on $\mathbb{R}$, each of these functions vanishes outside a finite interval) such that these sequences converge to f a.e. on E.